SAMIKHIYA A Multidisciplinary Research Journal
Volume-1 (01), Oct. 2022, pp.77-93
Available at :https://journal.mscw.ac.in/Research_Journal/latest_issues.aspx

SPECIAL TYPES OF MATRICES AND THEIR APPLICATIONS

Harneet Kaur Matharoo[1], Rashmi Verma[2], Salvi Jain[3] and Sonia Aneja[4]
[1]harneetkaurmatharoo@gmail.com ,
2rashmiv710@gmail.com , 3salvijain1612@gmail.com , 4soniaanand@hotmail.com

Department of Mathematics, Mata Sundri College for Women, University of Delhi, Delhi

ABSTRACT

Matrix theory is an important field of study with wide scope of research. The applications of matrices range from scientific domains to real life problems, being used either directly or through numerical and geometric analysis. This survey article presents an overview about some of the special types of matrices, their properties and real-life applications.

Keywords: Matrices, Determinants, Matrix operations.

1. INTRODUCTION

The term matrix was first introduced by James Joseph Sylvester in 19th century followed by the algebraic aspects of matrices presented by Arthur Cayley (Cayley, 1858) which gave a boost to matrix theory and developed into a recognized and interesting field of research. Over the years, with the evolution of the concept, matrices have seen an extended use in research, commerce, social science and are being used in computer graphics, optics, cryptography, economics, chemistry, geology, robotics and animation, wireless communication and signal processing and finance.

Matrices have a long history of use in solving linear equations, dating back to 300BC. Nine Chapters of the Mathematical Art (Yong and Suanshu, 1994), a key Chinese literature, presents a method to solve simultaneous linear equations along with an idea of determinant and rectangular arrays. Dating from 300 BC to 200 AD, the literature contains the first documented example of the use of matrices to solve simultaneous linear equations. The method to solve simultaneous linear equations was introduced to Europe by the Italian mathematician Gerolamo with the publication of Ars Magna (Cardano and Spon ,1965) in 1545. The determinant which was introduced by the Japanese mathematician Seki Takakazu in 1683 first appeared in Chinese literatur e over 2,000 years ago. Chapter 8 viz., Methods of rectangular arrays, shows matrix-like number arrangements along with a method for solving simultaneous equations using a counting board which is mathematically identical to Carl Friedrich Gauss's modern matrix method (Jamil, 2012) of solution.

The Method of Solving Dissimulated Problems, published in 1683, by Seki Takakazu established a generic method for determining the determinant of a matrix and using it to solve equations. In his work Elements of Curves, published in 1659, Dutch mathematician Jan de Witt used arrays to depict transformations. In the first decade of 18^{th} century, Gottfried Wilhelm Leibniz popularized the use of arrays for storing information or solving problems. In 1750, Cramer introduced Cramer's rule which provides generic formula defined as determinants for any unknown in linear equation system. At the turn of the nineteenth century, Gauss proposed a strategy for solving a system of linear equations without introducing the

concept of matrices which is now known as Gauss Elimination method (Jamil, 2012; Dopico, 2012). However, it was in 1850 that the term matrix was introduced and the matrix theory developed a strong base in mathematics. In 1850, James Joseph Sylvester used the term matrix, which comes from the Latin word womb, to describe a collection of numbers. Sylvester (1851) defined a matrix as a rectangular array of terms, out of which different systems of determinants may be engendered, as from the womb of a common parent; these cognate determinants being by no means isolated in their relations to one another, but subject to certain simple laws of mutual dependence and simultaneous deperition. In simple terms, a matrix is a rectangular array which includes numbers, symbols, or expressions, arranged in rows and columns. Each cell in the matrix is called an element or an entry.

The use of arrays in early matrix theory was almost solely limited to determinants and Cayley's abstract matrix operations were ground-breaking. Matrix multiplication and matrix algebra is a result of Arthur Cayley's work. Two papers, one in 1850 and the other in 1858 published Cayley's work which includes the inverse of the matrix, rules for matrix compounding which is same as matrix multiplication and some more properties of matrices. He was a pioneer in the development of a matrix notion that was independent of equation systems. Cayley devised and demonstrated the well-known Cayley–Hamilton theorem in his memoir on the theory of matrices published in 1858, which states that every square matrix satisfies its own characteristic equation.

Matrix theory emerged as an important mathematical theory due to the work of female mathematician Olga Taussky Todd (1977) popularly known as the mother of matrix theory, who used matrices to investigate vibrations on airplanes during World War II. She developed several concepts viz., number theoretical integral matrices, integral matrices connected with number theory, etc. The modern bracket notation of matrices was introduced by Cuthbert Edmund Cullis in 1913. He was the first one to illustrate the use of $A = a_{i,j}$ to represent a matrix where $a_{i,j}$ refers to the element in the ith row and the jth column. In 1955, Leonid Mirsky's text, An Introduction to Linear Algebra played a significant role in making matrix theory as one of the most important areas of research.

Matrices have a wide range of applications in science and have been applied to solve real-world problems. Matrices are used to represent real-world data, message encryption and decryption, cryptography, coding theory, creating 3-D images and 2-D motion, to compress electronic data and to store fingerprint data, robotics and automation, CT scans and MRI scans, in economics to calculate gross domestic products, wireless application protocol, profit prediction, UV spectroscopy, automobiles, etc. The matrices are used in physics while applying Kirchhoff's Laws of Voltage and Current to solve problems, to explore electrical circuits, quantum mechanics and optics, to create graphs, calculate statistics, and conduct scientific research in a variety of domains.

In this paper, section 2 presents 5 special types of matrices viz., Hadamard matrix, Sparse matrix, Idempotent matrix, Vandermonde matrix and Pascal matrix. In section 3 we discuss Rhotrix and its properties. Section 4 concludes the paper.

2. SPECIAL TYPES OF MATRICES

In this section, we study about few of the special types of matrices, their properties and applications. Our focus in this survey will be on Hadamard matrix, Sparse matrix, Idempotent matrix, Vandermonde matrix and Pascal matrix.

2.1. Hadamard Matrix

Hadamard matrix (Hedayat and Wallis, 1978) is defined as follows:

Definition 2.1.1 : Hadamard matrix is a square matrix of order n with entries either +1 or - 1 whose rows and columns are pairwise orthogonal.

For example: $H_1 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$ is a Hadamard matrix.

It is worth noting that every matrix with entries +1 and -1 may not be Hadamard matrix.

For example: $H_2 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$ is not a Hadamard matrix.

The matrix H_2 has entries +1 and -1 but the rows and columns are not mutually orthogonal. The third row does not give dot product as zero with any other rows, and hence is not orthogonal to them. Similarly, the second column does not give dot product as zero with any other columns and is also not orthogonal.

A condition for an $n \times n$ Hadamard matrix to exist is that n must be 1, 2 or a multiple of 4. In 1867, Sylvester constructed Hadamard matrix of order 1, 2,4,8,16,32, etc. and later on matrices of order 12 and 20 were constructed by Hadamard in 1893. Hadamard matrix is named so as the determinant of the matrix satisfies equality in the Hadamard's determinant theorem. Sylvester formed a Hadamard matrix of order 2^k for all non-negative integer k as follows:

$$H_{2^k} = \begin{bmatrix} H_{2^{(k-1)}} & H_{2^{(k-1)}} \\ H_{2^{(k-1)}} & -H_{2^{(k-1)}} \end{bmatrix}$$

which is known as Sylvester-Hadamard matrix. Another construction of Hadamard matrix is Paley construction which was given by Raymond Paley in 1933. Earlier, the combination of Sylvester's and Paley's method did not produce matrix of order 92. Baumert, Golomb, and Hall constructed a matrix of order 92 using a computer in 1962. In 2005, a Hadamard matrix of order 428 was discovered for the first time by Hadi Kharaghani and Behruz Tayfeh-Rezaie [34]. As of 2014, there is no Hadamard matrix known for 668, 716, 892, 1132, 1244, 1388, 1436, 1676, 1772, 1916, 1948, and 1964 which are 12 multiples of 4 less than or equal to 2000.

In a Hadamard matrix, $HH^T = nI$ where H^T is the transpose of H, and I is the identity matrix. This implies H is non-singular and has inverse $n^{-1}H^T$. Hadamard matrix satisfies Hadamard's determinant inequality which states that if $A = a_{i,j}$ is a matrix of order n where $|a_{ij}| \leq 1$ for all i and j, then $|\det A| \leq n^{n/2}$. In a Hadamard matrix, if any row or column is multiplied by -1, then the matrix remains Hadamard matrix.

A Hadamard matrix is called a Skew Hadamard matrix if $H + H^T = 2I$. Just like Hadamard matrix, Skew Hadamard matrix exists for every n divisible by 4. As of 2006, it was known that Skew Hadamard matrix exist for all $n < 188$ with n divisible by 4.

Hadamard matrix is used to retrieve data from communication systems, digital signal systems and image encoding in the presence of disturbances which rely on statistical approaches. Sylvester Hadamard matrices are used in a variety of fields including statistics, numerical analysis, coding theory, and cryptography. In 1951, the satellite Plotkin18 was the first to demonstrate the ability of codes derived from Hadamard matrices to repair errors. This was further investigated by Bose and Shrikhande in 1959, who established a link between Hadamard matrices and block code designs, and then developed by Harmuth in 1960 (Seberry et.al., 2005). Levenshstein was the first who constructed an algorithm for Hadamard Error Correcting Code. The Hadamard codes are simple to decode, and they are the first significant class of codes to repair multiple errors. A famous application of Hadamard Error Correcting Code was in the NASA space mission when the Mariner and Voyager space probes explored Mars and the outer planets of the solar system from 1969 to 1976 and they utilised a Hadamard code to encode data.

2.2. Sparse Matrix

A sparse matrix or sparse array is a matrix with most of its components as zero. The sparsity of a matrix is calculated by dividing the number of zero-valued elements by the total number of elements. A sparse matrix (Schneider and Taussky, 1977)is defined as follows:

Definition 2.2.1 : A matrix in which most of its elements are zero is called a sparse matrix. For example: $\begin{bmatrix} 0 & 0 & 4 & 0 & 8 \\ 0 & 0 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 9 & 5 & 0 & 0 \end{bmatrix}$ is a sparse matrix.

In order to use resources effectively, we only store non-zero elements instead of storing zero's with them. Triplets (Row, Column, value) are used to store non-zero elements. There are two types of sparse matrix representations viz., array representation and linked list representation. In an Array representation, only non-zero values, as well as their row and column index values are considered in formulation. The total number of rows, total number of columns, and total number of non-zero values in the sparse matrix are all stored in the 0th row in this representation. For example,

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 7 & 0 \\ 0 & 6 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 8 \\ 0 & 0 & 5 & 0 & 0 & 0 \end{bmatrix} \rightarrow$$

Rows	Columns	Values
5	6	6
0	4	7
1	1	6
2	0	3
2	3	4
3	5	8
4	2	5

Figure 1: A 5 x 6 matrix

A linked list data structure is used to represent a sparse matrix in linked representation. We employ two different nodes in this linked list: the header node and the element node (Figure 2). Three fields make up the header node, while five fields make up the element node.

Header Node

IndexValue	
Down	Right

Element Node

Row	column	Value
down/up		Right

Figure 2: Header node and element node in linked list

H_0, H_1..., H_n are the header nodes that are used to represent indices, where *n* is the number of rows of the matrix. Except for the first node, which is used to convey abstract information of the sparse matrix, the remaining nodes are utilised to represent non-zero items in the matrix.

We use Sparse matrix rather than a simple matrix because there are fewer non-zero elements than zeros and as a result less memory is utilised to store only elements. By rationally creating a data structure that only traverses non-zero components, computing time can be decreased.

In combinatorics and applications such as network theory and numerical analysis, where crucial data or connections are frequently sparse, the concept of sparsity is useful. When solving partial differential equations, large sparse matrices are frequently used in scientific and engineering applications. Sparse matrices are common in machine learning, hence specialised computers have been created for them. Sparse matrices can be found in data preparation encoding techniques, structural engineering, reservoir simulation**,** electrical networks and in optimizing problems.

2.3. Idempotent Matrix

In 1870, Benjamin Peirce introduced the term idempotent. An idempotent matrix is defined as follows:

Definition 2.3.1 : A matrix A is idempotent if and only if $A^2 = A$.

For example: $\begin{bmatrix} 1 & -2 \end{bmatrix}$ and $\begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$ are idempotent matrices.

Since for an idempotent matrix A, $A^2 = A$, therefore A must be a square matrix. Idempotent matrices are also known as periodic matrices or n-potent matrices (i.e., $A^n = A$). In particular, if n=3, A is referred to as a tripotent matrix, and if n=4 it is referred to as a quadripotent matrix. For a 2 x 2 idempotent matrix either the matrix is diagonal or has a trace equal to 1. Every idempotent matrix is singular except the identity matrix. If an idempotent matrix is subtracted from an identity matrix of same order, then the resultant matrix is also an idempotent matrix. A non-identity matrix is idempotent if the number of independent rows (and columns) is less than the number of rows in the matrix (and columns).

R.E.Hartwig and M.S.Putcha (1990) identified a set of criteria under which a matrix can be expressed as a sum of idempotent matrices or as a difference of two idempotent matrices. Any complex matrix T is a sum of finitely many idempotent matrices, as demonstrated by Pei Yuan Wu (1990), if and only if *trace* T is an integer and $tr\ T \geq rank\ T$. J.K.Baksalary and O.M.Baksalary (2000) found a complete solution to the problem of describing all cases in which linear combinations of two different idempotent matrices P_1 and P_2 are also idempotent matrices. Every tripotent matrix B can be decomposed in a unique way as $B = B_1 - B_2$, where B_1 and B_2 are idempotent matrices such that $B_1B_2 = B_2B_1 = 0$.

Idempotent matrix is useful in functional analysis, particularly in the spectrum theory of transformations and projections. They are often used in regression analysis, econometrics and the theory of linear statistical models.

2.4. Vandermonde Matrix

Vandermonde matrix is a special type of matrix in linear algebra named after the French mathematician Alexandre-Théophile Vandermonde. Vandermonde did not discuss the concept of a Vandermonde matrix but gave the underlying principle of Vandermonde determinant or specifically, a method to solve linear equations in (Vandermonde, 1776) which was further shaped and presented by Sylvester (1850, 1851) and Cayley (Rawashdeh, 2019) after a century. A Vandermonde matrix (Strang, 2016) is defined as follows:

Definition 2.4.1 : Vandermonde matrix is a simple $m\ x\ n$ matrix with entries of the rows or columns as geometric progressions. It is of the form,

$$V_{mn}(x_n) = \begin{bmatrix} 1 & x_1 & {x_1}^2 & \cdots & {x_1}^{m-1} \\ 1 & x_2 & {x_2}^2 & \cdots & {x_2}^{m-1} \\ 1 & x_3 & {x_3}^2 & \cdots \vdots & {x_3}^{m-1} \\ \vdots & \vdots & & \ddots & \vdots \\ 1 & x_n & \cdots & \cdots & {x_n}^{m-1} \end{bmatrix}$$

and can be written as $V_{ij} = x_i^{j-1}$, $\forall\ i\ and\ j$, where x_i's are called nodes or points.

For example: $V_1 = \begin{bmatrix} 1 & 1 & 1^2 & 1^3 \\ 1 & 2 & 2^2 & 2^3 \\ 1 & 3 & 3^2 & 3^3 \\ 1 & 4 & 4^2 & 4^3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix}$ is a Vandermonde matrix.

Vandermonde determinant or Vandermonde polynomial (or Vandermondian) (Muir, 1906; Vandermonde,1776) is the determinant of a square Vandermonde matrix (n=m) which is the product of all differences of the values that define the elements and is given by,

$v_n(x_1, \ldots, x_n) = \det(V_n(x_1, \ldots, x_n)) = \det(V) = \prod_{1 \le i < j \le n} (x_j - x_i)$.

Over the period of time, various proofs for the Vandermonde determinant were given using combinatorics or using induction along with elementary row or column operations (Mirsky, 1955; Knuth ,1997) or graph based techniques (Gessel, 1979). The determinant is non-singular or non-zero if and only if $x_i \neq x_j, \forall\, i$ and j, i.e., all x_i's should be distinct and thus the square Vandermonde matrix is invertible where inverse is given by (Soto-Eguibar, and Moya-Cessa, , 2011)

$$(V_n^{-1})_{ij} = \frac{(-1)^{j-1}\sigma_{n-j,i}}{\prod_{\substack{k=1 \\ k \neq i}}^{n} (x_k - x_i)}$$

where, $\sigma_{j,i} = \sum_{1 \le m_1 < m_2 < \ldots < m_j \le n} \prod_{k=1}^{j} x_{m_k}(1 - \delta_{m_k,i})$ and, $\delta_{a,b} = \begin{cases} 1, & a = b \\ 0, & a \neq b \end{cases}$.

Different researchers have given different methods of calculating the inverse of a Vandermonde matrix. Yiu (2014) showed inverse as a product of two matrices, one of them being a lower triangular matrix using the partial fraction based technique, and F. Soto and H. Moya

(2011) showed inverse as a product of diagonal, upper triangular and lower triangular matrices. According to L. Richard (1966), inverse of the matrix is written as a product of simpler matrices, namely, upper triangular and lower triangular matrix.

If A^{-1} represents inverse of a Vandermonde matrix, then it can be written as: $A^{-1} = U^{-1} L^{-1}$, where, U is an upper triangular matrix and L is a lower triangular matrix.

$$U^{-1} = \begin{bmatrix} 1 & -x_1 & x_1x_2 & -x_1x_2x_3 & \cdots \\ 0 & 1 & -(x_1+x_2) & x_1x_2 + x_2x_3 + x_3x_1 & \cdots \\ 0 & 0 & 1 & -(x_1+x_2+x_3) & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \text{ and,}$$

$$L^{-1} = \begin{bmatrix} 1 & 0 & 0 & \cdots \\ \frac{1}{x_1-x_2} & \frac{1}{x_2-x_1} & 0 & \cdots \\ \frac{1}{(x_1-x_2)(x_1-x_3)} & \frac{1}{(x_2-x_1)(x_2-x_3)} & \frac{1}{(x_3-x_1)(x_3-x_2)} & \cdots \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}.$$

Many attempts at generalizing the Vandermonde matrix have been made which is of the form:

$$G_{mn}(x_n) = [x_j^{\alpha_i}]_{i,j}^{m,n} = \begin{bmatrix} {x_1}^{\alpha_1} & {x_1}^{\alpha_2} & \cdots & {x_1}^{\alpha_m} \\ {x_2}^{\alpha_1} & {x_2}^{\alpha_2} & \cdots & {x_2}^{\alpha_m} \\ {x_3}^{\alpha_1} & {x_3}^{\alpha_2} & \cdots & {x_3}^{\alpha_m} \\ \vdots & \vdots & \ddots & \vdots \\ {x_n}^{\alpha_1} & \cdots & \cdots & {x_n}^{\alpha_m} \end{bmatrix}$$

where $x_i \in \mathbb{C}$, $\alpha_i \in \mathbb{C}, i = 1,2, \ldots, n$. One such generalization is the Alternant matrix (Strang, 2016) which is obtained by replacing the geometric progressions by functions. It is represented as,

$$A_{mn}(f_m; x_n) = [f_i(x_j)]_{i,j}^{m,n} = \begin{bmatrix} f_1(x_1) & f_1(x_2) & f_1(x_3) & \cdots & f_1(x_n) \\ f_2(x_1) & f_2(x_2) & f_2(x_3) & \cdots & f_2(x_n) \\ f_3(x_1) & f_3(x_2) & f_3(x_3) & \cdots & f_3(x_n) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f_m(x_1) & f_m(x_2) & \cdots & \cdots & f_m(x_n) \end{bmatrix},$$

where $f_1, f_2, \ldots, f_m$ are functions from a field $F \rightarrow F$.

Karl Lundengård (Strang, 2016) mentioned some of the varied applications of the Vandermonde matrix ranging from polynomial interpolation , optimal experiment design, least square regression, error-detecting and error-correcting codes construction, to calculation of the discrete Fourier and related transforms , solving systems of differential equations with constant coefficients , various problems in mathematical, nuclear, and quantum physics and describing properties of the Fisher information matrix of stationary stochastic processes. One of the simplest method used to find the interpolating polynomial is using the Vandermonde matrix, which under generalized conditions can also be seen for interpolation of non-polynomial functions or points in more than two dimensions. Vandermonde matrix can also be used to prove the unisolvence theorem.

3. RHOTRIX

In this section we discuss rhotrices which is similar to matrices yet has a different structure. In 2002, Ajibade (2002) introduced rhotrix as an extension to the mathematical arrays which are somewhat in between two-dimensional vectors and 2 x 2 dimensional matrices, an extension to the idea of matrix-tertions and noitrets and forms a group in itself. The arrangement of entries of the rhotrix is rhomboidal. Study of rhotrix theory has two approaches: commutative ring theory and non-commutative ring theory. The commutative ring theory uses the heart based multiplication method as proposed by Ajibade (2002). This method is based on the analysis and initial algebra of rhotrices. The non-commutative ring theory uses a row-column based method for rhotrix multiplication as proposed by Sani (2004).

Definition 3.1 : A rhotrix is defined as follows:

$$A = R(3) = \left\{ \left\langle \begin{array}{ccc} & a & \\ b & h(A) = c & d \\ & e & \end{array} \right\rangle : a, b, c, d, e \in \mathbb{R} \right\} \quad (1)$$

It consists of mainly 3 parts - heart, major and minor entries. Here, *h(A) = c* is called the heart of the rhotrix *A* defined as the element at the perpendicular intersection of the two diagonals of a rhotrix.

According to Ajibade (2002) if we consider *A* and *B* to be two three-dimensional rhotrices, then addition (+), scalar multiplication and multiplication (◦) of these rhotrices is defined as:

$$A = \left\{ \left\langle \begin{array}{ccc} & a_1 & \\ b_1 & h(A) = c_1 & d_1 \\ & e_1 & \end{array} \right\rangle : a_1, b_1, c_1, d_1, e_1 \in \mathbb{R} \right\}$$ and

$$B = \left\{ \left\langle \begin{array}{ccc} & a_2 & \\ b_2 & h(B) = c_2 & d_2 \\ & e_2 & \end{array} \right\rangle : a_2, b_2, c_2, d_2, e_2 \in \mathbb{R} \right\} \quad (2)$$

$$A + B = \left\langle \begin{array}{ccc} & a_1 + a_2 & \\ b_1 + b_2 & h(A) + h(B) & d_1 + d_2 \\ & e_1 + e_2 & \end{array} \right\rangle. \quad (3)$$

For any scalar $\alpha \in \mathbb{R}$: $\alpha A = \left\langle \begin{array}{ccc} & \alpha a_1 & \\ \alpha b_1 & \alpha h(A) & \alpha d_1 \\ & \alpha e_1 & \end{array} \right\rangle$

$$A \circ B = \left\langle \begin{array}{ccc} & a_1 h(B) + a_2 h(A) & \\ b_1 h(B) + b_2 h(A) & h(A)h(B) & d_1 h(B) + d_2 h(A) \\ & e_1 h(B) + e_2 h(A) & \end{array} \right\rangle.$$

This method of rhotrix multiplication is referred to as Ajibade's heart based method for multiplication of rhotrices. A similar row-column based method for rhotrix multiplication was given by Sani [36] as,

$$A \circ B = \left\langle \begin{array}{ccc} & a_1 a_2 + d_1 b_2 & \\ b_1 a_2 + e_1 b_2 & h(A)h(B) & a_1 d_2 + d_1 e_2 \\ & b_1 d_2 + e_1 e_2 & \end{array} \right\rangle.$$

The definition given by Ajibade was further extended to an n-dimensional rhotrix by Mohammed in [44]. A rhotrix of size n is given as:

$$A(n) = R(n) = \left\{ \left\langle \begin{array}{ccccccccc} & & & & a_1 & & & & \\ & & a_2 & & a_3 & & a_4 & & \\ & \cdots & \cdots & & \cdots & & \cdots & \cdots & \\ a_{\{\frac{t+1}{2}\}-n/2} & \cdots & \cdots & & h(A) = a_{\{\frac{t+1}{2}\}} & & \cdots & \cdots & a_{\{\frac{t+1}{2}\}+n/2} \\ & \cdots & \cdots & & \cdots & & \cdots & \cdots & \\ & & a_{t-3} & & a_{t-2} & & a_{t-1} & & \\ & & & & a_t & & & & \end{array} \right\rangle : a_1, a_2, \dots, a_t \in \mathbb{R} \right\} \quad (4)$$

where $n \epsilon 2\mathbb{Z}^+ + 1$, $t = \frac{1}{2}(n^2 + 1)$ and $h(A) = a_{\{\frac{t+1}{2}\}}$ is the heart of the rhotrix matrix in R(n).

There are two types of rhotrices: an even dimensional rhotrix and an odd dimensional rhotrix. An odd dimensional rhotrix is simply a rhotrix having odd dimensions, for eg. the simplest odd dimensional rhotrix is a 3-dimesional rhotrix we mentioned earlier (1). An even-dimensional rhotrix is a rhotrix having even dimensions. Simplest even dimensional rhotrix is defined as follows:

$$R(2) = \left\{ \left\langle \begin{array}{ccc} & a & \\ b & & d \\ & e & \end{array} \right\rangle : a, b, d, e \in \mathbb{R} \right\}.$$

This is known as a real rhotrix of dimension two, also known as heartless rhotrix (*hl*-rhotrix). A.O. Iser (2018) gave the cardinality of an even-dimensional rhotrix as the $|R(n)| = \frac{1}{2}(n^2 + 2n)$, $where\ n \epsilon\ 2\mathbb{N}$. Addition and multiplication for even dimensional and odd dimensional rhotrices are the same.

Mohammed (2007) defined rhotrix exponent rule for any integer *m* as,

$$R^m = (h(R))^{m-1} = \left\langle mb \quad \begin{matrix} ma \\ h(R) \\ me \end{matrix} \quad md \right\rangle.$$

If *m* = 0, then the identity of rhotrix is defined as:

$$\mathbf{I} = R^0 = \left\langle 0 \quad \begin{matrix} 0 \\ 1 \\ 0 \end{matrix} \quad 0 \right\rangle \tag{5}$$

and if *m* = -1, then the inverse of a rhotrix is given by:

$$R^{-1} = \frac{-1}{(h(R))^2} \left\langle b \quad \begin{matrix} a \\ -h(R) \\ e \end{matrix} \quad d \right\rangle. \tag{6}$$

For a rhotrix R with the defined rhotrix addition (4),$\langle R, + \rangle$ forms a commutative group with identity element and inverse defined as given by (5) and (6), respectively. A rhotrix A is said to be invertible if $h(A) \neq 0$, and if $h(A) = 0$ and $A \neq 0$ then $A^2 = 0$.

Similarly, Identity *hl*-rhotrix is defined as:

$$\mathbf{I} = \left\langle 0 \quad \begin{matrix} 1 \\ \\ 1 \end{matrix} \quad 0 \right\rangle$$

and inverse of a *hl*-rhotrix is defined as:

$R^{-1} = \frac{1}{ae-bd} \left\langle -b \quad \begin{matrix} e \\ \\ a \end{matrix} \quad -d \right\rangle$, where $ae \neq bd$.

Along with an n-dimensional rhotrix as (4), Mohammed (2007) also defined an n-dimensional rhotrix as follows:

$$R(n) = \left\langle \begin{array}{ccccccccc} & & & & a_{11} & & & & \\ & & & a_{21} & c_{11} & a_{12} & & & \\ & \cdots & & \cdots & \cdots & \cdots & \cdots & & \\ a_{t1} & \cdots & & \cdots & \cdots & \cdots & \cdots & & a_{tt} \\ & \cdots & & \cdots & \cdots & \cdots & \cdots & & \\ & & & a_{tt-1} & c_{t-1t-1} & a_{t-1t} & & & \\ & & & & a_{tt} & & & & \end{array} \right\rangle$$

where $a_{ij}(i,j = 1,2,\dots,t)$ are called the major entries and $c_{kl}(k,l = 1,2,\dots,t-1)$ are called the minor entries of R. A n-dimensional rhotrix R(n) = $\langle a_{ij}, c_{kl}\rangle$ is called as a couple of two matrices namely (a_{ij}) called the major matrix and (c_{kl}) called the minor matrix. A Hilbert rhotrix of size 5 was formed using a Hilbert matrix of size 3 x 3 and of size 2 x 2 using this concept. We can also define rank of a rhotrix R(n) = $\langle a_{ij}, c_{kl}\rangle$, where $a_{rr}(1 \le r \le t)$ and $c_{ss}(1 \le s \le t-1)$ as, rank (R) = rank (a_{ij}) + rank (c_{kl}), where properties of rank of matrix can be extended to rank of rhotrix.

Rhotrix theory has applications in algebra. Using the exponent rule and the properties of rhotrices [43,42], the algebraic series and expansion can be expressed in terms of rhotrices. If we consider the rhotrices A and B (2), the arithmetic series in terms of rhotrices is given by:

$$S_n = \sum_{k-1}^{n}[A + (n-1)B]$$

where S_n is the arithmetic progression with A as the first term and B as the common difference. The n^{th} sum is given as:

$$S_n = \frac{n}{2}\{2A + (n-1)B\} = \frac{n}{2}\left\{2\left\langle \begin{array}{ccc} & a_1 & \\ b_1 & h(A) & d_1 \\ & e_1 & \end{array}\right\rangle + (n-1)\left\langle \begin{array}{ccc} & a_2 & \\ b_2 & h(B) & d_2 \\ & e_2 & \end{array}\right\rangle\right\}.$$

$$\Rightarrow S_n = \left\langle \begin{array}{ccc} & \frac{2na_1+n(n-1)a_2}{2} & \\ \frac{2nb_1+n(n-1)b_2}{2} & \frac{2nh(A)+n(n-1)h(B)}{2} & \frac{2nd_1+n(n-1)d_2}{2} \\ & \frac{2ne_1+n(n-1)e_2}{2} & \end{array}\right\rangle.$$

Similarly, we can define the Geometric series in terms of rhotrices as follows:

$$U_n = \sum_{k=1}^{n}[A \circ B^{k-1}]$$

where A is the first term and B is the common ratio. The n^{th} sum is given as:

$U_n = A \circ (\frac{I-B^n}{I-B})$.

$\Longrightarrow U_n = A \circ \{(I-B^n) \circ (I-B)^{-1}\}, when\ h(B) \neq 1.$

The polynomial equations with variables and coefficients as rhotrices are called the rhotrix polynomial equations. If A and B are the real rhotrices as defined in (2), then the equations $X + A = B$ and $C \circ Y = D$ have unique solutions~~, g~~iven by $X = B - A$ and $Y = D \circ C^{-1}$, respectively, when $h(C) \neq 0$.

4. CONCLUSION

Matrix theory as a whole is a valuable and useful concept in the field of engineering, physics, economics, statistics and various branches of mathematics. Determinant of a matrix was found and used much before the idea of matrix as an algebraic entity emerged and thus the determinant of a matrix is a very practical application. In this article, we focused on some special forms of matrix and properties which make them different yet a powerful tool for solving real life problems.

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